Calculations of Eigenvalues in Functional Nonlinear Spinor Theory *

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A differential equation of third order for spinor potentials is proposed, that modifies the dynamics of the nonlinear spinor theory. We derive a symmetrical eigenvalue equation using functional integration techniques. This equation and a momentum symmetrized equation — a simplified form of the mass eigenvalue equation proposed by Stumpf — are applied to calculate mass eigenvalues. By a special combination of both methods it is possible to weaken the regularization dipole in Heisenberg's theory and thereby produce better boson masses. Finally, the modified theory allows a self-consistent calculation of the fermion propagator.

1. Introduction

If canonical quantum theory is applied to relativistic elementary particle physics, it leads to the well known divergence difficulties. To avoid these divergences HEISENBERG [1,2] introduced an indefinite metric in the space of physical states. Hence the metrical structure of that space is not known from the beginning, but will emerge from the solution of the theory. Consequently, the field operators which act in this unknown space cannot be well-defined objects. Therefore to formulate the theory only the transition matrix elements are left, or their generating functionals.

In functional quantum theory, as established and described in detail by STUMPF in several papers [3,4,5], one tries to calculate all physical information of a quantum theory with the aid of functionals, meaning that not only eigenvalues have to be computed using stationary functionals, but especially the S-matrix has to be expressed in terms of scattering functionals. For that purpose one not only has to formulate the dynamical equation in functional space, but all the additional conditions as well representing the subsidiary conditions for the definition of quantum numbers for stationary functionals and the asymptotic conditions for scattering functionals including their normalization. This program has already been carried through to a great extend. The calculation of the S-matrix is the main purpose of functional quantum theory. But the eigenvalue kernels occur in the scattering functionals too, therefore it is convenient to study eigenvalue equations first.

We will not only discuss Heisenberg's well known nonlinear spinor theory, but a dynamically modified theory as well, which seems to be more suitable for later scattering calculations and which allows a self-consistent determination of the fermion propagator. In chapter 2 the new equation of third order for spinor potentials is proposed.

In chapter 3 symmetrized eigenvalue equations are derived by means of an indefinite functional integration. This procedure avoids the use of a special conserved quantity and nevertheless allows the formal solution of the functional equations. Thus, one is able to formulate approximation methods in an elegant way. In chapter 4 the lowest approximations are calculated explicitly. The corresponding approximations of a simplified form of the mass eigenvalue equation proposed by STUMPF [6] are also calculated. Both methods are applied to Heisenberg's form of the nonlinear spinor theory too. In this case a special combination allows the weakening of the regularization dipole to obtain better boson masses.

In Chapter 5 the fermion propagator of the modified theory is calculated selfconsistently. Besides some technical details in the appendix a reformulation of both theories is sketched in terms of canonical fields.

2. A Dynamically Modified Spinor Theory

In Heisenberg's unified field theory of elementary particles [2] a nonlinear spinor equation of first order is postulated. The vacuum twopoint function F is regularized by a dipole of mass zero. We will refer to this as dipole theory. The modified theory will be regularized by a simple pole of mass zero. We will call it pole theory.

Similar to some recent papers on nonlinear spinor theory [7-10] we interpret the basic field as spinor potential. Differentiating this potential we generate physical fields which satisfy canonical commutation relations. Therefore the spinor potential itself satisfies noncanonical commutation relations and it becomes necessary to introduce an indefinite metric in the space of physical states. Up to now a differential equation of first order was postulated for this spinor potential. In this case the theory is scale invariant at small distances and hence formally renormalizable. Its renormalization has been studied in lowest order perturbation theory [11]. A proof of the renormalizability in each order, however, seems to be very complicated, because of the two kinds of propagators involved. Therefore the topological structure of the graphs is more intricate than in quantum electrodynamics.

To avoid those difficulties in the present paper a differential equation of third order for the spinor potential is postulated. Like Heisenberg's dipole theory the new pole theory is superrenormalizable, thus no divergences occur in perturbation calculations. Because of the third time derivation not only does the vanishing equal-time-commutator appear in the functional equations, but the equal-time-commutator of the derivatives — i.e. the physical fields — which does not vanish (Appendix 1). Therefore in the pole theory one is able to calculate the fermion propagator (Chapter 5). This is possible in spite of the indefinite metric indicating that it is more harmless here than in the dipole case.

For we may rewrite the pole theory in terms of the canonical physical fields alone which then satisfy a nonlocal differential equation. Such a canonical formulation is possible in the dipole theory too, but there the canonical ϱ_0-term is subsequently compensated by the more nonlocal interaction, thus one cannot calculate the fermion propagator in an easy way. In the pole theory such compensation does not occur (App. 1).

The group theoretical structure of the pole theory is completely taken from Heisenberg's dipole theory [2]. Consequently we consider two-component Weyl spinors as a representation of the Lorentz group, because only then a unique local interaction term is possible, whereas with four-component Dirac spinors the well known five interaction terms are allowed. It is true that each Dirac theory can be formulated with Weyl spinors too [12], but only the

$(v-a)$-coupling of the Dirac theory remains local in Weyl's formulation. The other four combinations yield derivative couplings. In another paper I shall come back to this point in more detail, there I will discuss scattering functionals for incoming free massive particles within the framework of Weyl's formulation.

To describe isotopic spin the number of components has to be doubled [13]. For technical reasons we will deal with a Hermitean representation of the spinors, therefore we have to double the number of components once more [14]. We will use the notation of Dürr and Wagner for the representation matrices.

It is possible to describe the pole and dipole theory simultaneously. Common statements are written without special indices, special forms are marked with a "p" resp. "d" for the pole resp. dipole theory. Especially we use the function

$$r := \begin{cases} r^p = 1 & \text{for the pole theory,} \\ r^d = 0 & \text{for the dipole theory.} \end{cases} \quad (1)$$

Both theories are defined by functional equations for the respective τ-functionals $|\mathfrak{T}(j)\rangle$

$$|\mathfrak{T}(j)\rangle := \sum_{n=0}^{\infty} \frac{i^n}{n!} \tau_{\alpha_1 \ldots \alpha_n} |j_{\alpha_1} \cdots j_{\alpha_n}\rangle. \quad (2)$$

The existence of the anticommuting sources j is guaranteed by explicit construction in a functional representation space [4] which shall not be mixed up with the space of physical states. To simplify the notation we will include the dependence on the coordinates also in the index. Repeated indices hence always indicate a summation over spinor indices and a 4-dimensional integration over the corresponding space-time variable.

In the usual operator formalism (App. 1) the τ-functions $\tau_{\alpha_1 \ldots \alpha_n}$ are matrix elements of time-ordered products of field operators between the vacuum and the state considered. One derives a functional equation for the τ-functional from an operator field equation and commutation relations. In functional quantum theory the functional equation itself serves as the definition of the theory:

$$D_{\alpha\beta}\, \partial_\beta \,|\, \mathfrak{T}(j)\rangle = \{i\, r\, \varrho_0\, j_\alpha - V_{\alpha\beta\gamma\delta}(\partial_\beta\, \partial_\gamma\, \partial_\delta \\ + 3\, F_{\beta\gamma}\, \partial_\delta)\}\,|\, \mathfrak{T}(j)\rangle. \quad (3)$$

∂_β denotes the functional derivative $\delta/\delta j_\beta$, r is the function defined in (1). The vertex operator V does not contain derivatives.

$$V_{\alpha\beta\gamma\delta} = \frac{K^2}{2}\, \delta(x_\alpha - x_\beta)\, \delta(x_\alpha - x_\gamma)\, \delta(x_\alpha - x_\delta)\, \bar{V}_{\alpha\beta\gamma\delta} \quad (4)$$

where $\overline{V}_{\alpha\beta\gamma\delta}$ does not depend on the space co-ordinates. $\overline{V}$ is explicitly given by Dürr and Wagner[14]. The $D_{\alpha\beta}$ is a Hermitean, purely imaginary antisymmetrical differential operator of third order in the pole theory, of first order in the dipole theory.

$$D_{\alpha\beta} := i\, \Gamma^{\nu}_{\alpha\beta}\, \delta(x_\alpha - x_\beta)\, \frac{\partial}{\partial x^{\nu}_{\beta}} \left[-\frac{\partial}{\partial x_{\beta}{}^{\mu}} \frac{\partial}{\partial x_{\beta\mu}} \right] \tag{5}$$

with $\Gamma^{\nu}_{\alpha\beta}$ a symmetrical and real matrix which transforms like a 4-vector under Lorentz transformations.

The vacuum twopoint function or fermion propagator has the form

$$F_{\alpha\beta} := \langle 0\,|\,T\,\psi_\alpha\,\psi_\beta\,|\,0\rangle \tag{6}$$
$$= i \int \mathrm{d}m^2 \, \varrho\,(m^2) \int \frac{d^4p}{(2\,\pi)^4} \frac{\overline{\Gamma}^{\nu}_{\alpha\beta}\, p_\nu}{(p^2 - m^2)\,(p^2)^{2-r}} \, e^{-i(p,x_\alpha - x_\beta)}.$$

$\overline{\Gamma}^{\nu}$ transforms with the conjugate representation of Γ^{ν}. One has

$$\overline{\Gamma}^{\nu}_{\alpha\beta}\, p_\nu\, \Gamma^{\mu}_{\beta\gamma}\, p_\mu = \delta_{\alpha\gamma}\, p^2, \qquad \overline{\Gamma}^{0}_{\alpha\beta} = \Gamma^{0}_{\alpha\beta} = \delta_{\alpha\beta}. \tag{7}$$

In the explicit form of Dürr and Wagner is

$$\Gamma^{\mu}_{\alpha\beta} := -[\lambda'_2\, \sigma'^{\mu}]_{\alpha\beta}, \qquad \overline{\Gamma}^{\mu}_{\alpha\beta} := -[\lambda'_2\, \overline{\sigma}'^{\mu}]_{\alpha\beta}. \tag{8}$$

In F we omitted the usual factor m^{4-2r} in the numerator of the integral (6). Then the fields have the natural (length-)dimension defined by the singularity of F for equal arguments.

$$\dim \psi = +\tfrac{1}{2} - r. \tag{9}$$

The coupling constants K^2 in both theories hence have the dimension $(\text{mass})^2$ as usual in superrenormalizable theories.

With the functional Eq. (3) the pole theory is defined in the main. One has to clarify only how to treat the regularization pole. Like the dipole in Heisenberg's theory the pole is supposed to belong to no physical particles. Hence no imaginary parts should start at the corresponding thresholds. You may suppress them by taking the principle value at the $|\boldsymbol{p}|$-integration[2], or by taking a suitable limit from imaginary masses towards zero[15,16], or by calculating all integrals with the Feynman $i\,\varepsilon$-prescription and cancelling all "wrong" imaginary parts afterwards[16]. We will use the last method without answering the question how far it will be possible to interpret such a procedure physically, as it has been done in the dipole case in lowest approximations[2,17].

The question is now to solve Eq. (3). First of all we invert the differential operator D with the aid of the Green's function

$$G_{\alpha\beta} := i\, D^{-1}_{\alpha\beta} = i \int \frac{d^4p}{(2\,\pi)^4} \frac{\overline{\Gamma}^{\nu}_{\alpha\beta}\, p_\nu}{(p^2)^{1+r}} \, e^{-i(p,\,x_\alpha - x_\beta)}, \tag{10}$$

$$\partial_\alpha\,|\,\mathfrak{T}(j)\rangle = \{r\,\varrho_0\, G_{\alpha\beta}\, j_\beta + i\, G_{\alpha\alpha'}\, V_{\alpha'\beta\gamma\delta}\,(\partial_\beta\,\partial_\gamma\,\partial_\delta + 3\,F_{\beta\gamma}\,\partial_\delta)\}\,|\,\mathfrak{T}(j)\rangle. \tag{11}$$

We neglect possible inhomogeneous terms since at present we consider stationary functionals only. Therefore it does not matter that we had to invert a differential operator of third order. For scattering functionals one has to examine the inhomogeneous terms carefully. Like the mass zero pole in F the mass zero pole in G shall correspond to no physical particles. In higher approximations one pole will be dressed to a finite physical mass.

In the next chapter we will invert the functional differential operator too. Before doing that we introduce the usual φ-functional which is supposed to be more suitable for approximations[2,18]. Its connection to the τ-functional is expressed by the Wick rule as used in conventional perturbation theory

$$|\varPhi(j)\rangle := e^{\frac{1}{2}j_\alpha F_{\alpha\beta} j_\beta}\,|\,\mathfrak{T}(j)\rangle = \sum_{n=0}^{\infty} \frac{i^n}{n!}\, \varphi_{\alpha_1\ldots\alpha_n}\,|\,j_{\alpha_1}\ldots j_{\alpha_n}\rangle. \tag{12}$$

The φ-functions are the τ-functions in which all twopoint functions F are subtracted out. From (11) we get the equation for $|\varPhi(j)\rangle$

$$\partial_\alpha\,|\,\varPhi(j)\rangle = \mathfrak{D}_\alpha(j,\partial)\,|\,\varPhi(j)\rangle:$$
$$= \{(r\,\varrho_0\, G_{\alpha\beta} - F_{\alpha\beta})\, j_\beta + i\, G_{\alpha\alpha'}\, V_{\alpha'\beta\gamma\delta}\,(d_\beta\, d_\gamma\, d_\delta + 3\, F_{\beta\gamma}\, d_\delta)\}\,|\,\varPhi(j)\rangle \tag{13}$$

with the notation

$$d_\alpha := \partial_\alpha + F_{\alpha\beta}\, j_\beta. \tag{14}$$

To solve (13) one has to ask for subsidary conditions which restrict the manifold of solutions and fix the quantum numbers of the physical state con-

sidered. For example the momentum condition reads for a state with momentum I^μ

$$\hat{p}^\mu |\Phi\rangle = I^\mu |\Phi\rangle \qquad (15)$$

with the momentum operator in functional space

$$\hat{p}^\mu := (j_\alpha\, p^\mu_{\alpha\beta}\, \partial_\beta)\,, \qquad p^\mu_{\alpha\beta} := (1/i)\, \delta_{\alpha\beta}\, \delta/\delta x_{\beta\mu}. \qquad (16)$$

We do not need explicitly the corresponding operators for other quantum numbers [19].

3. Symmetrical Approximation by a Functional Integration

3.1. *The average problem*

The functional Eq. (13) singles out one co-ordinate in the corresponding φ-system (App. 2). The exact φ-functions as well as the τ-function are (anti-)symmetrical in all variables, hence all approximations should be symmetrical too and the co-ordinates should all be treated the same. Therefore some averaging procedure is necessary. The obvious average

$$(j_\alpha\, \partial_\alpha)\,|\Phi\rangle = (j_\alpha\, \mathfrak{D}_\alpha)\,|\Phi\rangle \qquad (17)$$

is not satisfactory, because you cannot invert the operator $(j\,\partial)$ without some trouble. One has

$$(j\,\partial)\,|j^n\rangle = n\,|j^n\rangle \qquad (18)$$

and $(j\,\partial)$ is not the identity. For later applications we are looking after the following form of (13)

$$|\Phi\rangle = \mathfrak{E}\,|\Phi\rangle \qquad (19)$$

to formulate functional approximation methods.

One possibility is to use a conserved quantity for an averaging method, especially the mass [6]. From (15) we get for a state with momentum I^μ

$$I^\mu\,|\Phi\rangle = (j\,p^\mu\,\partial)\,|\Phi\rangle = (j\,p^\mu\,\mathfrak{D})\,|\Phi\rangle \qquad (20)$$

and hence for this mass average with $m^2 = I^\mu\,I_\mu$

$$|\Phi(j)\rangle = \frac{1}{m^2}\,I^\mu\,I_\mu\,|\Phi\rangle = \frac{1}{m^2}\,(j\,p^\mu\,\mathfrak{D})\,I_\mu\,|\Phi\rangle$$
$$= \frac{1}{m^2}\,(j\,p^\mu\,\mathfrak{D})\,(j\,p_\mu\,\mathfrak{D})\,|\Phi\rangle =: \mathfrak{E}^m\,|\Phi\rangle. \qquad (21)$$

For explicite calculations this $\mathfrak{E}^m$ has the disadvantage, that it is quadratic in $\mathfrak{D}$, hence the lowest approximation to (21) contains an iteration of (13) already. Therefore in the lowest boson calculation already you have the task of angular momentum reduction. This complicated problem for vector bosons is treated by SCHEERER [20]. For scalar bosons

there is simpler way. In this paper we use instead of the mass average $\mathfrak{E}^m$ essentially the first line of (21) i.e. a momentum average

$$|\Phi\rangle = \frac{1}{I^2}\,I^\mu\,(j\,p_\mu\,\mathfrak{D})\,|\Phi\rangle =: \mathfrak{E}^q\,|\Phi\rangle. \qquad (22)$$

We use $1/I^2$ instead of $1/m^2$ for later convenience. The integrals then have the same asymptotic behavior as in the case of the integral average derived in Section 3.3. At an eigenvalue is $I^2 = m^2$ in any case.

But this momentum average too is sometimes not very suitable, because it creates additional powers of momenta in Feynman integrals. First of all the evaluation of those integrals becomes more complicated, but they may even diverge in some cases. You will find examples in Chapter 4 during the explicit calculations. So we are looking after another average method.

3.2. *An average by functional integration*

The simplest way to average (13) is the direct inversion of the functional differentiation. It can be done by an indefinite functional integration, which has been used by the author to develop functional recursion formulae in the case of the anharmonic oscillator [21].

Functional integrals which map functionals on real numbers require careful considerations about a suitable integral measure. Such integrals are necessary to answer questions of convergence of approximation methods and have been studied by many authors [22]. We need an indefinite functional integral which maps functionals on functionals. We define this integral as the inverse of the functional differentiation

$$\int \delta j_\alpha\,|\,\mathfrak{F}_\alpha(j)\rangle = |\,\mathfrak{S}(j)\rangle :\Leftrightarrow \partial_\alpha\,|\,\mathfrak{S}(j)\rangle = |\,\mathfrak{F}_\alpha(j)\rangle. \qquad (23)$$

That means, our integral is not an infinite dimensional volume integral but an infinite dimensional line integral.

The main properties of this integral follow immediatly from (23)

$$\partial_\beta \int \delta j_\alpha\,|\,\mathfrak{F}_\alpha(j)\rangle = |\,\mathfrak{F}_\beta(j)\rangle, \qquad (24)$$

$$\int \delta j_\alpha\,\partial_\alpha\,|\,\mathfrak{S}(j)\rangle = |\,\mathfrak{S}(j)\rangle + \text{const}. \qquad (25)$$

Other properties are deduced from respective properties of the functional differentiation, e.g. the linearity

$$\int \delta j_\alpha\,|\,a\,\mathfrak{F}_\alpha + b\,\mathfrak{G}_\alpha\rangle = a \int \delta j_\alpha\,|\,\mathfrak{F}_\alpha\rangle + b \int \delta j_\alpha\,|\,\mathfrak{G}_\alpha\rangle \qquad (26)$$

or the rule of partial integration

$$\int \delta j_\alpha \, |\, \mathfrak{S}_1 (\partial_\alpha \mathfrak{S}_2)\rangle = |\, \mathfrak{S}_1 \mathfrak{S}_2\rangle - \int \delta j_\alpha \, |\, (\partial_\alpha \mathfrak{S}_1)\, \mathfrak{S}_2\rangle \,. \quad (27)$$

In the framework of functional quantum theory the existence of this indefinite integral is guaranteed, because we may expand each functional in terms of the basic functionals $|j^n\rangle$. The remaining integrals over power functionals are evaluated with

$$\int \delta j_{\alpha_0} f_{\alpha_0 \ldots \alpha_n} |\, j_{\alpha_1} \ldots j_{\alpha_n}\rangle = \frac{1}{n+1} \, f_{\alpha_0 \ldots \alpha_n} |\, j_{\alpha_0} \ldots j_{\alpha_n}\rangle + \text{const}\,. \quad (28)$$

Moreover, we are never to evaluate an integral explicitly. In all applications there is at least one functional differentiation on the left hand side of each integral, thus all integrals drop out again with (24). Hence we are allowed to omit the free integration constant in the following equations.

With the aid of the new integral (23) we get from (13)

$$|\, \Phi(j)\rangle = \int \delta j_\alpha \, \mathfrak{D}_\alpha \, |\, \Phi(j)\rangle =: \mathfrak{E}^1 \, |\, \Phi(j)\rangle\,. \quad (29)$$

The j in (28) anticommute, hence from f the antisymmetrical part only is left over. Therefore the integral (23) serves as an average procedure and it yields the symmetrization of the φ-function system (see App. 2).

In Chapter 4 eigenvalues are calculated with this integral average $\mathfrak{E}^1$ and the momentum average $\mathfrak{E}^q$. Naturally we have to approximate (29) resp. (23). The form of this eigenvalue equation allows a simple formulation of approximation methods.

3.3. *Approximation methods*

Let us define projection operators $\mathbf{P}_n$ in functional space, projecting on the basic power functionals[5].

$$\mathbf{P}_n := n! \, |\, D_n(\alpha_1 \ldots \alpha_n)\rangle \langle D_n(\alpha_1 \ldots \alpha_n)|\,. \quad (30)$$

The basic states are given by

$$|\, D_n(\alpha_1 \ldots \alpha_n)\rangle := \frac{1}{n!} \, j_{\alpha_1} \ldots j_{\alpha_n} |\, 0\rangle\,, \quad (31)$$

$$\langle D_n(\alpha_1 \ldots \alpha_n)| := |\, D_n(\alpha_1 \ldots \alpha_n)\rangle^+ \quad (32)$$

with the orthonormalization properties

$$\langle D_n(\alpha_1 \ldots \alpha_n) \,|\, D_m(\beta_1 \ldots \beta_m)\rangle \quad (33)$$

$$= \delta_{nm} \frac{1}{(m!)^2} \sum_{\mathrm{p}} (-)^{\mathrm{p}} \, \delta(\alpha_1 - p(\beta_1)) \ldots \delta(\alpha_m - p(\beta_m))\,.$$

For the projection operators (30) we have now

$$\mathbf{P}_m \,|\, D_n(\alpha_1 \ldots \alpha_n)\rangle = \delta_{mn} \,|\, D_n(\alpha_1 \ldots \alpha_n)\rangle\,, \quad (34)$$

$$\mathbf{P}_m \mathbf{P}_n = \delta_{mn} \mathbf{P}_n\,. \quad (35)$$

For each set of quantum numbers there is a smallest ϱ with $\varphi_\varrho \neq 0$ and $\varphi_\nu = 0$ for $\nu < \varrho$, e.g. for a state of baryon number N, spin S and isospin I we have $\varrho = \max(|N|,\, 2S,\, 2I)$. For the φ-functional of this state we have

$$\mathbf{P}_\nu \,|\, \Phi\rangle = 0\,, \qquad \nu = 0, \ldots, \varrho - 1\,, \quad (36)$$

$$|\, \Phi_\varrho\rangle := \mathbf{P}_\varrho \,|\, \Phi\rangle \neq 0\,. \quad (37)$$

We denote the sum of the higher projection operators with

$$\mathbf{M}_\varrho := \sum_{\nu=\varrho+1}^{\infty} \mathbf{P}_\nu\,. \quad (38)$$

Now we are able to write the eigenvalue Eq. (19) as an equation for $|\, \Phi_\varrho\rangle$ alone[5]

$$|\, \Phi_\varrho\rangle = \mathbf{P}_\varrho \, \mathfrak{E}(|\, \Phi_\varrho\rangle + \mathbf{M}_\varrho \,|\, \Phi\rangle)\,, \quad (39)$$

$$\mathbf{M}_\varrho \,|\, \Phi\rangle = \mathbf{M}_\varrho \, \mathfrak{E}(|\, \Phi_\varrho\rangle + \mathbf{M}_\varrho \,|\, \Phi\rangle)\,. \quad (40)$$

Solving (40) formally we get

$$\mathbf{M}_\varrho \,|\, \Phi\rangle = [1 - \mathbf{M}_\varrho \, \mathfrak{E}]^{-1} \mathbf{M}_\varrho \, \mathfrak{E} \,|\, \Phi_\varrho\rangle \quad (41)$$

and finally

$$|\, \Phi_\varrho\rangle = \mathbf{P}_\varrho \, \mathfrak{E}[1 - \mathbf{M}_\varrho \, \mathfrak{E}]^{-1} \,|\, \Phi_\varrho\rangle\,. \quad (42)$$

We have to solve (42) together with respective subsidiary conditions for the discrete quantum numbers to obtain mass eigenvalues.

Of course the problem is the construction of the inverse operator in (42). The simplest method consists of an expansion in Neumann's series

$$|\, \Phi_\varrho\rangle = \mathbf{P}_\varrho \, \mathfrak{E} \sum_{\nu=0}^{\infty} (\mathbf{M}_\varrho \, \mathfrak{E})^\nu \,|\, \Phi_\varrho\rangle\,. \quad (43)$$

That leads to an expansion in powers of the coupling constant, in which the exact twopoint function F occurs.

In analogy to the NTD-method[2] we may neglect in the system $(39) - (40)$ all $\mathbf{P}_\nu \,|\, \Phi\rangle$, $\nu \geqq N$ for a certain N and may try to solve this truncated system.

In this paper we make use of the first nonvanishing term of (43) for numerical calculations. Corresponding approximations have been used in nonlinear spinor theory most of all[2]. Indeed, the integral average in the dipole theory leads to the well known results.

4. Calculation of Mass Eigenvalues

We approximate the mass spectral function $\varrho(m^2)$ of the fermion propagator F (6) by $\delta(m^2 - \varkappa^2)$, then F has a pole at the nucleon mass $\varkappa$. We will came back to the selfconsistency of this assumption in Chapter 5. If we denote Fourier transforms by a tilde "$\sim$", we obtain from (6)

$$\widetilde{F}_{\alpha\beta} = i\,\overline{\varGamma}^{\nu}_{\alpha\beta}\,p_{\nu}/[(p^2 - \varkappa^2)(p^2)^{2-r}] =: \overline{\varGamma}^{\nu}_{\alpha\beta}\,F_{\nu}(p). \quad (44)$$

4.1. *Fermion mass eigenvalue equation*

We will derive an equation for the nucleons first. With $S = \frac{1}{2}$, $I = \frac{1}{2}$, $B = 1$ we get $\varrho = 1$ and from (43)

$$|\varPhi_1\rangle = \mathbf{P}_1\,\mathfrak{E}\,|\varPhi_1\rangle + \mathbf{P}_1\,\mathfrak{E}\,\mathbf{M}_1\,\mathfrak{E}\,|\varPhi_1\rangle + \cdots \quad (45)$$

The first term vanishes, hence we have to deal with the second term to get a nontrivial equation. Applying $\langle D_1| \triangleq (\delta/\delta j)|_{j=0}$ to (45) it becomes an

equation for the one point φ-function φ^{F} of the fermion, which reads for the integral average $\mathfrak{E}^{\mathrm{i}}$ in configuration space

$$\varphi^{\mathrm{F}}_{\alpha} = 6\left(-i\,\frac{K^2}{2}\right)^2 G_{\alpha\beta}\,V^i_{\beta\gamma}\,G_{\gamma\delta}\,V^j_{\delta\mu}\,F_{\mu\nu}\,V^i_{\nu\lambda}\,F_{\lambda\varrho}\,V^j_{\varrho\sigma}\,\varphi^{\mathrm{F}}_{\sigma}. \quad (46)$$

In (46) we have already comprehended three terms (A21) using the antisymmetry of V.

The interaction matrices (4) are written in the form

$$V^i_{\alpha\beta}\,V^i_{\gamma\delta} := \overline{V}_{\alpha\beta\gamma\delta} \quad (47)$$

given explicitly in the paper of Dürr and Wagner[14]. In momentum space we get from (46) and from a respective Eq. (A21) for the momentum average $\mathfrak{E}^{\mathrm{q}}$ with (44) and (10) in usual matrix notation without spin indices

$$\tilde{\varphi}^{\mathrm{F}}(I) = -6\left(\frac{K}{2\pi\varkappa}\right)^4 \overline{\varGamma}_{\lambda}\,V^i\,\overline{\varGamma}_{\mu}\,V^j\,\overline{\varGamma}_{\nu}\,V^i\,\overline{\varGamma}_{\varrho}\,V^j\,\tilde{\ }^{\mathrm{F}}(I)$$

$$\cdot\left[\frac{I^{\alpha}I_{\alpha}}{I^2}\right]^{\mathrm{e}} G^{\lambda}(I)\,\frac{\varkappa^4}{(2\pi)^4}\int \mathrm{d}^4r\,\mathrm{d}^4s\left[3\,\frac{I^{\beta}(I+r-s)_{\beta}}{I^2}\right]^{\mathrm{e}} G^{\mu}(I+r-s)\,F^{\nu}(r)\,F^{\varrho}(s). \quad (48)$$

This equation holds for integral and momentum average (22) resp. (29) with the aid of the function

$$\mathrm{e} := \begin{cases} \mathrm{e}^{\mathrm{i}} = 0 \text{ for integral average}, \\ \mathrm{e}^{\mathrm{q}} = 1 \text{ for momentum average}. \end{cases} \quad (49)$$

The momentum average has an additional factor $I^{\alpha}p_{\alpha}/I^2$ at each Green's function $G(p)$. The factor 3 follows from the momentum condition (15) (see App. 2). The spin matrices in (48) are evaluated to give

$$\overline{\varGamma}_{\lambda}\,V^i\,\overline{\varGamma}_{\mu}\,V^j\,\overline{\varGamma}_{\nu}\,V^i\,\overline{\varGamma}_{\varrho}\,V^j = -\frac{16}{3}\,\overline{\varGamma}_{\lambda}(\varGamma_{\mu}\,g_{\nu\varrho} + \varGamma_{\nu}\,g_{\varrho\mu} + \varGamma_{\varrho}\,g_{\mu\nu}), \quad (50)$$

$$= -\frac{16}{3}\,(g_{\lambda\mu}\,g_{\nu\varrho} + g_{\lambda\nu}\,g_{\varrho\mu} + g_{\lambda\varrho}\,g_{\mu\nu}). \quad (51)$$

The last equation holds, if (50) is applied to symmetrical expressions in $(\lambda\mu\nu\varrho)$. We may use it since the integral in (48) depends on I^{μ} only. We obtain

$$\left(1 + \frac{3}{2}\left(\frac{K}{2\pi\varkappa}\right)^4 L(\lambda)\right)\tilde{\varphi}^{\mathrm{F}}(I) = 0, \quad \lambda := I^2/\varkappa^2, \quad (52)$$

$$L(\lambda) := \frac{1}{3}\,(g_{\lambda\mu}\,g_{\nu\varrho} + g_{\lambda\nu}\,g_{\varrho\mu} + g_{\lambda\varrho}\,g_{\mu\nu})\,\frac{I^{\lambda}}{(I^2)^{1+r}}\,L^{\mu\nu\varrho}, \quad (53)$$

$$L^{\mu\nu\varrho} := \frac{\varkappa^4}{\pi^4}\int \mathrm{d}^4r\,\mathrm{d}^4s\left[3\,\frac{(I, I-r-s)}{I^2}\right]^{\mathrm{e}}\frac{(I-r-s)^{\mu}\,r^{\nu}\,s^{\varrho}}{[(I-r-s)^2]^{1+r}[r^2 - \varkappa^2][r^2]^{2-r}[s^2 - \varkappa^2][s^2]^{2-r}}. \quad (54)$$

The lengthy calculation of the integrals is sketched in Appendix 3, where analytic expressions for the corresponding functions $L(\lambda)$ are derived.

We will use the eigenvalue Eq. (52) to determine the coupling constant K^2. For that purpose we identify the mass $\varkappa^2$ of the fermion propagator F

with the fermion mass eigenvalue I^2 in (52) and obtain

$$(K/2\pi\varkappa)^4 = 2/(-3L(1)). \tag{55}$$

The numerical results are discussed in Section 4.3 and 4.4 for the pole resp. dipole theory.

4.2. *Boson mass eigenvalue equation*

The simplest boson equation results from (42) with $\varrho = 2$

$$|\Phi_2\rangle = \mathbf{P}_2\,\mathfrak{F}\,|\Phi_2\rangle + \cdots. \tag{56}$$

In this case the first term already yields a nontrivial equation. Applying $\langle D_2| \triangleq (\delta^2/\delta j^2)|_{j=0}$ to (56) it becomes an equation for the twopoint φ-function φ^B of the boson which reads for the integral average $\mathfrak{F}^i$ in configuration space (A 22)

$$\varphi^{\mathrm{B}}_{\alpha\beta} = -3\left(-i\,\frac{K^2}{2}\right)G_{\alpha\beta'}\,V^i_{\beta'\gamma}\,F_{\gamma\beta}\,V^i_{\mu\nu}\,\varphi^{\mathrm{B}}_{\nu\mu}. \tag{57}$$

If we define the Fourier transform $\tilde{\varphi}^{\mathrm{B}}_{\alpha\beta}$ by

$$\varphi^{\mathrm{B}}_{\alpha\beta} = e^{-i\frac{1}{2}(I,\,x_\alpha+x_\beta)}\int \frac{d^4p}{(2\pi)^4}\,\tilde{\varphi}^B(p,I)\,e^{-i(p,\,x_\alpha-x_\beta)} \tag{58}$$

we obtain from (57) and from a corresponding Eq. (A 22) for the momentum average $\mathfrak{F}^q$ in matrix notation

$$\tilde{\varphi}^B(p,I) = \frac{3}{2}\left(\frac{K}{2\pi\varkappa}\right)^2\frac{i\varkappa^2}{4\pi^2}\,\overline{T}_\alpha\,V^i\,\overline{T}_\beta\,\mathrm{Tr}[V^i\int d^4s\,\tilde{\varphi}^B(s,I)]\left[2\,\frac{(I,\,p+I)}{I^2}\right]^e G^\alpha(p+I)\,F^\beta(p). \tag{59}$$

Performing an integration over p we obtain an algebraic equation

$$\varphi^B(I) := \int d^4p\,\tilde{\varphi}^B(p,I) = \frac{3}{2}\left(\frac{K}{2\pi\varkappa}\right)^2\overline{T}_\alpha\,V^i\,\overline{T}_\beta\,\mathrm{Tr}[V^i\,\varphi^B(I)]\frac{i\varkappa^2}{4\pi^2}\int d^4p\left[2\,\frac{(I,\,p+I)}{I^2}\right]^e G^\alpha(p+I)\,F^\beta(p). \tag{60}$$

To fix the discrete quantum numbers the functional $|\Phi_2\rangle$ has to satisfy the corresponding subsidiary conditions. In our simple case of the two point function, however, it is convenient to introduce projection operators P^{B}_{SI} (B = baryon number, S = spin, I = isospin) for φ_2 itself, which project on corresponding states [14].

$$P^{\mathrm{B}}_{SI\,\alpha\beta,\gamma\delta}\,P^{\mathrm{B}'}_{S'I'\,\gamma\delta,\varkappa\lambda} = \delta_{BB'}\,\delta_{SS'}\,\delta_{II'}\,P^{\mathrm{B}}_{SI\,\alpha\beta,\varkappa\lambda} \tag{61}$$

If we introduce the boson functions $q_\nu(\lambda)$, $\lambda := I^2/\varkappa^2$,

$$i\,\frac{4\varkappa^2}{\pi^2}\int d^4p\left[2\,\frac{(I,\,I-p)}{I^2}\right]^e\frac{(I-p)^\alpha p^\beta}{[(I-p)^2]^{1+r}[p^2-\varkappa^2][p^2]^{2-r}} =: \left(\frac{1}{2}\,g^{\alpha\beta} - \frac{I^\alpha I^\beta}{I^2}\right)q_0(\lambda) + \left(\frac{1}{2}\,g^{\alpha\beta} + \frac{I^\alpha I^\beta}{I^2}\right)q_1(\lambda), \tag{62}$$

$$q_2(\lambda) := \tfrac{1}{2}\,q_0(\lambda) + \tfrac{3}{2}\,q_1(\lambda) \tag{63}$$

we may write (60) in the form

$$\varphi^B(I) = \left(\frac{K}{2\pi\varkappa}\right)^2\frac{1}{4}\,[(3\,P^0_{00} + P^0_{01})\,q_0 + (3\,P^0_{10} + P^0_{11})\,q_1 - 2\,P^2_{01}\,q_2]\,\varphi^B(I). \tag{64}$$

The calculation of the four integrals (63) is sketched in App. 3, where we derive analytic expressions for the q_ν-functions. We will use (64) for numerical calculations of boson masses in the pole and dipole theory.

In neither case we find solutions for $B = 2$. We expected that, because for meaningfull deuteron solutions the Green's function G has to be dressed to a physical mass first. Solutions for the vector bosons ($S = 1$) are to reject, since the corresponding physical states would have negative norm because of the positive derivative of $q_1(\lambda)$ at the eigenvalue [2,23]. Therefore we are left with the scalar bosons ($S = 0$). We will calculate the mass of the η-singlett ($I = 0$) and of the π-triplett ($I = 1$) which is degenerate because of the isospin invariance of our theories.

4.3. *Results for the pole theory* ($r = 1$)

The integral average $\mathfrak{F}^i$ yields the fermion function $L^{\mathrm{pi}}(\lambda)$ from (53) with

$$L^{\mathrm{pi}}(\lambda) = \frac{1}{6}\ln|\lambda| + \frac{(1-\lambda)^3}{3\lambda^3}\ln|1-\lambda| - \frac{4-\lambda}{3\lambda}\sqrt{\frac{4-\lambda}{\lambda}}\arcsin\frac{\sqrt{\lambda}}{2} + \frac{1}{3\lambda^2} \tag{65}$$

for $0 \leq \lambda \leq 4$. For other values of λ see (A 37).

The asymptotic expansions for $\lambda \to 0$ and $\lambda \to \pm \infty$ are given by

$$L^{\mathrm{pi}} = -\frac{1}{2\lambda} + \frac{\ln|\lambda|}{6} - \frac{1}{6} + O(\lambda)$$
$$= \frac{1}{\lambda^2} + \frac{\ln\lambda}{\lambda^3} + O(\lambda^{-3}). \quad (65')$$

Especially we have

$$L^{\mathrm{pi}}(1) = \frac{1}{3} - \frac{\sqrt{3}}{6}\pi = -0.5736. \quad (66)$$

Therefrom we obtain the coupling constant of the pole theory

$$(K^{\mathrm{p}}/2\pi\varkappa)^2 = 1.078. \quad (67)$$

The momentum average cannot be applied to the fermion calculation, since the integral (54) diverges logarithmically for $r = 1$, $e = 1$.

The boson functions q_0, q_1 for the integral average are

$$q_0^{\mathrm{pi}}(\lambda) = \ln|\lambda| + \frac{3-\lambda^2}{\lambda^2}\ln|1-\lambda| + \frac{3}{\lambda}$$
$$= \ln|\lambda| - \frac{3}{2} + O(\lambda^2) \quad (68)$$
$$= \frac{4}{\lambda} + \frac{3}{\lambda^2}\ln|\lambda| + \frac{1}{2\lambda^2} + O(\lambda^{-3}),$$

$$q_1^{\mathrm{pi}}(\lambda) = \ln|\lambda| - \frac{1+\lambda^2}{\lambda^2}\ln|1-\lambda| - \frac{1}{\lambda}$$
$$= \ln|\lambda| + \frac{1}{2} + \frac{4}{3}\lambda + O(\lambda^2) \quad (69)$$
$$= -\frac{1}{\lambda^2}\ln|\lambda| + \frac{1}{2\lambda^2} + O(\lambda^{-3}).$$

We obtain with (55) the boson masses

$$m_\pi^{\mathrm{pi}} = 0.33\varkappa, \qquad m_\eta^{\mathrm{pi}} = 1.04\varkappa. \quad (70)$$

λ_π is small in all cases, hence we may calculate m_π with sufficient accuracy from the asymptotic expansion of q_0:

$$q_0(\lambda) \approx \ln|\lambda| - c \quad (71)$$

leading to

$$m_\pi \approx \varkappa \cdot \exp\left[\frac{c}{2} - 2\left(\frac{2\pi\varkappa}{K}\right)^2\right]. \quad (72)$$

The experimental values are for $\varkappa =$ nucleon mass

$$m_\pi^{\exp} = 0.147\varkappa, \qquad m_\eta^{\exp} = 0.585\varkappa. \quad (73)$$

The calculated masses (70) are too big, since q_0^{pi} has apart from the singularity at $\lambda = 0$ another one at $\lambda = 1$. Therefore q_0^{pi} is too small in the important region (see Fig. 1).
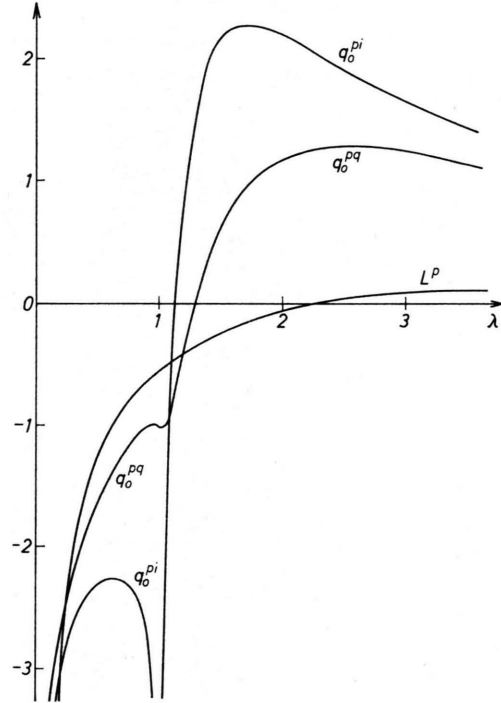

Fig. 1. Pole theory: Fermion function $L^{\mathrm{p}}(\lambda)$ and boson functions $q_0^{\mathrm{p}}(\lambda)$ of the scalar bosons. q_1^{pi} for integral average, q_0^{pq} for momentum average.

The corresponding q_0^{pq} function of the momentum average does not have the singularity at $\lambda = 1$ because of the additional factor $(I, I-q)/I^2$ in (62). Hence q_0^{pq} is larger than q_0^{pi} (Fig. 1) and we expect better mass eigenvalues.

We calculate the q^{pq}-functions for the momentum average

$$q_0^{\mathrm{pq}}(\lambda) = \ln|\lambda| - \frac{1}{\lambda^3}(4-\lambda-\lambda^2)(1-\lambda)\ln|1-\lambda| - \frac{4}{\lambda^2} + \frac{3}{\lambda}$$
$$= \ln|\lambda| - \frac{7}{6} + \frac{\lambda}{3} + O(\lambda^2) = \frac{4}{\lambda} + \frac{5}{\lambda^2}\ln|\lambda| - \frac{7}{2\lambda^2} - \frac{4}{\lambda^3}\ln|\lambda| + O(\lambda^{-3}), \quad (74)$$

$$q_1^{\mathrm{pq}}(\lambda) = \frac{1}{3}\ln|\lambda| + \frac{1}{3\lambda^3}(4+\lambda+\lambda^2)(1-\lambda)\ln|1-\lambda| + \frac{4}{3\lambda^2} - \frac{1}{3\lambda}$$
$$= \frac{1}{3}\ln|\lambda| + \frac{1}{18} + \frac{\lambda}{3} + O(\lambda^2) = \left(-\frac{1}{\lambda^2} + \frac{4}{3\lambda^3}\right)\ln|\lambda| + \frac{3}{2\lambda^2} + O(\lambda^{-3}) \quad (75)$$

and obtain the boson masses

$$m_\pi^{\mathrm{pq}} = 0.28\,\varkappa, \qquad m_\eta^{\mathrm{pq}} = 0.85\,\varkappa. \qquad (76)$$

The momentum average allows no calculation of the coupling constant in the pole case, but one obtains slightly better boson masses as discussed before.

4.4. Results for the dipole theory $(r = 0)$

The integral average yields from (54) with $e = 0$ the well known fermion function L of Heisenberg's theory[2] (A 39) with

$$q_0^{\mathrm{di}}(\lambda) = \ln|\lambda| - \frac{1}{\lambda^2}(1-\lambda)^2 \ln|1-\lambda| - \frac{1}{\lambda}$$

$$= \ln|\lambda| - \frac{3}{2} + \frac{\lambda}{3} + O(\lambda^2) = \left(\frac{2}{\lambda} - \frac{1}{\lambda^2}\right)\ln|\lambda| - \frac{3}{2\lambda^2} + O(\lambda^{-3}), \qquad (79)$$

$$q_1^{\mathrm{di}}(\lambda) = \left(1 - \frac{2}{3}\lambda\right)\ln|\lambda| + \frac{1}{3\lambda^2}(1+2\lambda)(1-\lambda)^2 \ln|1-\lambda| + \frac{1}{3\lambda} + \frac{2}{3}$$

$$= \left(1 - \frac{2}{3}\lambda\right)\ln|\lambda| + \frac{1}{2} + \frac{8}{9}\lambda + O(\lambda^2) = \frac{1}{3\lambda^2}\ln|\lambda| + \frac{1}{\lambda} + \frac{5}{18\lambda^2} + O(\lambda^{-3}). \qquad (80)$$

Hence we get the old result

$$m_\pi^{\mathrm{di}} = 0.31\,\varkappa, \qquad m_\eta^{\mathrm{di}} = 0.92\,\varkappa. \qquad (81)$$

The momentum average yields the functions

$$q_0^{\mathrm{dq}}(\lambda) = \ln|\lambda| + \frac{1}{\lambda^3}(1-\lambda)^3 \ln|1-\lambda| + \frac{1}{\lambda^2} - \frac{5}{2\lambda}$$

$$= \ln|\lambda| - \frac{11}{6} + \frac{\lambda}{4} + O(\lambda^2) = \left(\frac{3}{\lambda} + \frac{1}{\lambda^3} - \frac{3}{\lambda^2}\right)\ln|\lambda| - \frac{3}{2\lambda} - \frac{3}{2\lambda^2} + O(\lambda^{-3}), \qquad (82)$$

$$q_1^{\mathrm{dq}}(\lambda) = \left(\frac{5}{3} - \frac{2}{3}\lambda\right)\ln|\lambda| - \frac{1}{3\lambda^3}(1+2\lambda)(1-\lambda)^3 \ln|1-\lambda| - \frac{1}{3\lambda^2} + \frac{1}{6\lambda} + \frac{2}{3} \qquad (83)$$

$$= \ln|\lambda|\left(\frac{5}{3} - \frac{2}{3}\lambda\right) - \frac{7}{18} + \frac{41}{36}\lambda + O(\lambda^2) = \left(\frac{1}{\lambda} + \frac{1}{3\lambda^2} - \frac{1}{3\lambda^3}\right)\ln|\lambda| + \frac{3}{2\lambda} - \frac{13}{18\lambda^2} + O(\lambda^{-3}),$$

and if we use the coupling constant of the integral average

$$m_\pi^{\mathrm{dq}} = 0.36\,\varkappa, \qquad m_\eta^{\mathrm{dq}} = 1.06\,\varkappa. \qquad (84)$$

Thus the integral average yields exactly the results known from the old calculations. We expected this, since the integral average leads to the symmetrized φ-function system and we used the same approximation method. In the NTD-approximations you have to symmetrize the equations by hand[2], whereas with the new integral average only symmetrical equations arise. In the algebraic Eq. (52) and (60) of the lowest approximations this does not matter, since the corresponding momentum integrals are symmetrical anyhow.

We may understand the failure of the momentum average in the dipole case in the following way: F (6) and G (10) are very different $(r = 0!)$. For large

$$L^{\mathrm{di}}(1) = \frac{3\sqrt{3}}{16}\,\pi - \frac{\pi^2}{6} = -0.6247 \qquad (77)$$

and the corresponding coupling constant

$$(K^{\mathrm{di}}/2\,\pi\varkappa)^2 = 1.034. \qquad (78)$$

The momentum average yields another fermion function $L^{\mathrm{dq}}(\lambda)$ (A 40), but the corresponding $L^{\mathrm{dq}}(1)$ is positive and we get no real coupling constant.

For the boson calculation too we obtain for the integral average the well known q_ν-functions of Heisenberg's theory[2]

momenta we have $G(p) \approx (p^2)^2 F(p)$. The dipole may be a too strong regularization. This difference between F and G is even enlarged by the momentum average, since $G(p)$ is multiplied by another factor $(Ip)/I^2$ and in the equations occurs $p^\mu p^\nu / p^2$. The simple pole of G is further weakened. On the other hand you may expect better results by such a factor, if the dipole of F is weakened, i.e. if this factor occurs at F. Indeed, the numerical results of such an F-average show a correction to lower boson masses.

To obtain approximations with factors $(Ip)/I^2$ at $F(p)$ we pay attention to the equation

$$\frac{(I-q, I)}{I^2}\,G(I-q)\,F(q) \qquad (85)$$

$$+ G(I-q)\frac{(q, I)}{I^2}\,F(q) = G(I-q)\,F(q).$$

With (85) we find from (62) for the lowest boson functions

$$\tfrac{1}{2}(q^q + q^F) = q^i. \qquad (86)$$

Therefore the boson functions of the integral average are half the sums of the resp. functions of momentum and F-average. Since the momentum average gives larger boson masses than the integral average, the F-average gives smaller ones (see Fig. 2).

In functional form like (19) such an F-average may be obtained in the following somewhat artifical way. $|\Phi\rangle$ satisfies (22) and (29) and therefore a combination of both equations too

$$|\Phi\rangle = \{(1+\alpha)\,\mathfrak{E}^i - \alpha\,\mathfrak{E}^q\}\,|\Phi\rangle. \qquad (87)$$

Of course it is not in the spirit of a unified theory of elementary particles to introduce a new parameter. But for the special purpose of weakening the dipole it may be allowed. We get the F-average with (86) from (87)

$$|\Phi\rangle = (2\,\mathfrak{E}^i - \mathfrak{E}^q)\,|\Phi\rangle =: \mathfrak{E}^F\,|\Phi\rangle. \qquad (88)$$

The calculation of the corresponding fermion function L^{dF} (A 41) yields

$$L^{dF}(1) = -\frac{5}{4} + \frac{3\sqrt{3}}{8}\pi - \frac{\pi^2}{4} = -1.677. \qquad (89)$$

Therefore we get a real coupling constant

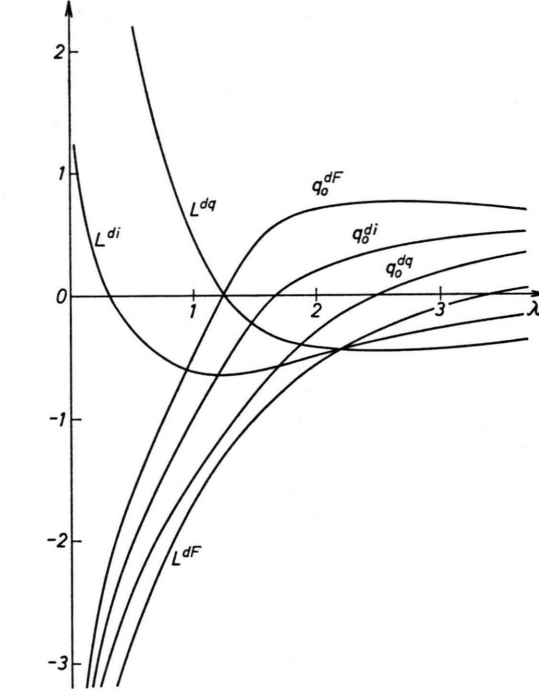$$(K^{dF}/2\pi\varkappa)^2 = 0.6305. \qquad (90)$$

Fig. 2. Dipole theory: Fermion functions $L^d(\lambda)$ and boson functions $q_0^d(\lambda)$ of the scalar bosons. "i" for integral average, "q" for momentum average, "F" for F-average with weakened dipole.

The corresponding boson functions are calculated from (86)

$$q_0^{dF}(\lambda) = \ln|\lambda| - \frac{1}{\lambda^3}(1+\lambda)(1-\lambda)^2 \ln|1-\lambda| - \frac{1}{\lambda^2} + \frac{1}{2\lambda}$$
$$= \ln|\lambda| - \frac{7}{6} + \frac{5}{12}\lambda + O(\lambda^2) = \left(\frac{1}{\lambda} + \frac{1}{\lambda^2} - \frac{1}{\lambda^3}\right)\ln|\lambda| + \frac{3}{2\lambda} - \frac{3}{2\lambda^2} + O(\lambda^{-3}), \qquad (91)$$

$$q_1^{dF}(\lambda) = \left(\frac{1}{3} - \frac{2}{3}\lambda\right)\ln|\lambda| + \frac{1}{3\lambda^3}(1+2\lambda)(1+\lambda)(1-\lambda)^2 \ln|1-\lambda| + \frac{1}{3\lambda^2} + \frac{1}{2\lambda} + \frac{2}{3}$$
$$= \left(\frac{1}{3} - \frac{2}{3}\lambda\right)\ln|\lambda| + \frac{25}{18} + \frac{23}{36}\lambda + O(\lambda^2) = \left(-\frac{1}{\lambda} + \frac{1}{3\lambda^2}\right)\ln|\lambda| + \frac{1}{2\lambda} + O(\lambda^{-2}), \qquad (92)$$

and we obtain the boson masses

$$m_\pi^{dF} = 0.08\varkappa, \qquad m_\eta^{dF} = 0.58\varkappa. \qquad (93)$$

Hence this F-average yields even a too small π-mass.

For comparison the calculated coupling constants and boson masses are shown in a table.

One has to prefer the average by means of the functional integral $\mathfrak{E}^i$ since this method is always

Table 1. Coupling constants and boson masses. The values in brackets are calculated with the coupling constant of the integral average.

theory	average method	$(K/2\pi\varkappa)^2$	$m_\pi/\varkappa$	$m_\eta/\varkappa$
pole	integral $\mathfrak{E}^i$	1.078	0.33	1.04
pole	momentum $\mathfrak{E}^q$	integral diverges	$[0.28]^i$	$[0.85]^i$
dipole	integral $\mathfrak{E}^i$	1.034	0.31	0.92
dipole	momentum $\mathfrak{E}^q$	imaginary	$[0.36]^i$	$[1.06]^i$
dipole	F-average $\mathfrak{E}^F$	0.630	0.08	0.58
experimental value			0.147	0.585

applicable, whereas the momentum average $\mathfrak{E}^q$ may lead to divergences even in a superrenormalizable theory. The F-average is only a specific tool handling the dipole case.

You should have in mind that the various average methods must lead to the same eigenvalues in higher approximations. The great differences therefore hints at the low accuracy of the approximations used.

5. Selfconsistent Calculation of the Fermion Propagator

We find an important difference between pole and dipole theory, if we do not consider eigenvalue equations only. There we found a similar behaviour, because the ϱ_0-term dropped out in our approximations. Calculating the fermion propagator, however, the ϱ_0-term becomes essential. In the pole theory it is present even with the indefinite metric (see App. 1). Therefore one is able to determine the fermion propagator out of the theory.

Really, one should calculate F from the η-functional which contains only the connected parts of the τ-functions[14], but in our approximations we may start from the φ-functional (12) for the vacuum state and use Eq. (13). We symmetrize (13) with the aid of the integral average (29), since we got a coupling constant with this method only. With the approximation used in our fermion calculation we obtain Eq. (46) again, in this case for F. The only difference is an additional ϱ_0-term which did not occur in (46) because of $\langle 0 | \text{Boson} \rangle = 0$.

$$F_{\alpha\tau} = r\varrho_0 G_{\alpha\tau} \qquad (94)$$
$$+ 6 \left(\frac{-iK^2}{2} \right) G_{\alpha\beta} V^i_{\beta\gamma} G_{\gamma\delta} V^j_{\delta\mu} F_{\mu\nu} V^i_{\nu\lambda} F_{\lambda\varrho} V^j_{\varrho\sigma} F_{\sigma\tau}.$$

With the notation of Sec. 4.1 this may be written

$$\widetilde{F} = r\varrho_0 \widetilde{G} - \frac{3}{2} \left(\frac{K}{2\pi\varkappa} \right)^4 L\widetilde{F}. \qquad (95)$$

In the pole theory is $r = 1$ and we obtain

$$\widetilde{F} = \frac{r\varrho_0 \widetilde{G}}{1 + \frac{3}{2} \left(\frac{K}{2\pi\varkappa} \right)^4 L}. \qquad (96)$$

In the dipole theory we would derive $F = 0$ from (96). One then has to extract another inhomogeneous term out of higher φ-functions[14] to replace $\varrho_0 G$ in (94). Moreover, one finds that in (95) both sides behave differently for large momenta in the dipol case.

To avoid this logarithmic inconsistencies the dipole regularization has to be replaced by a weaker one[9].

In the pole theory such problems do not occur. To calculate F from (96) we notice that L depends quadratically on F (94). Equation (96) hence is a complicated nonlinear equation for F, i.e. for the spectral function $\varrho(m^2)$ of F. As a first step of solving (96) we will consider it as a selfconsistence requirement. Therefore we replace $\varrho(m^2)$ by a δ-function at the nucleon mass again (44) and calculate the right hand side of (96) for later comparison with our ansatz (44). We obtain

$$F_R(p) = \frac{\varrho_0 \overline{\Gamma}^\nu p_\nu}{[p^2]^2 \left(1 + \frac{3}{2} \left(\frac{K}{2\pi\varkappa} \right)^4 L\left(\frac{p^2}{\varkappa^2} \right) \right)}. \qquad (97)$$

L is a short notation for L^{pi} of (65) resp. (A 37). This L yields the special values

$$L(1) = -0.5736, \quad L'(1) := \frac{\partial L}{\partial\lambda}\bigg|_{\lambda=1} = 0.8138,$$
$$L_0 := (\lambda L(\lambda))|_{\lambda=0} = -\frac{1}{2}, \quad L_\infty := (\lambda^2 L(\lambda))|_{\lambda=\infty} = 1. \qquad (98)$$

F_R has a pole for a finite p^2. We choose K^2 according to (55), then F_R has this pole for $p^2 = \varkappa^2$. We obtain the residuum at this pole by expansion of L

$$\text{Res } F_R(p^2 = \varkappa^2) = \frac{\varrho_0 \overline{\Gamma}^\nu p_\nu L(1)}{\varkappa^2(-L'(1))}. \qquad (99)$$

According to our ansatz (44) this residuum should be $\overline{\Gamma}^\nu p_\nu / \varkappa^2$. Comparing with (99) we obtain

$$\varrho_0 = L'(1)/(-L(1)) = 1.419. \qquad (100)$$

This ϱ_0 corresponds to the wave function renormalization constant Z_2^{-1} of a canonical theory, where Z_2 has to be less than 1[25]. Since we are able to formulate the pole theory in a canonical manner (App. 1), we had to expect (100). Because of the singularity of L for $\lambda = 0$ (98) F_R has a simple pole for $p^2 = 0$ which has a negative residuum.

$$\text{Res } F_R(p^2 = 0) = 1.628 \text{ Res } F(p^2 = 0). \qquad (101)$$

Finally, for large momenta F_R is proportional to F. No logarithmic inconsistencies do occur.

$$F_R(p^2 \to \infty) = 1.419 \, F(p^2 \to \infty). \qquad (102)$$

The calculated F_R hence shows a remarkable agreement with the ansatz (44), after we arranged the finite pole and the residuum at this pole. F_R not only reproduces both poles for $p^2 = \varkappa^2$, 0, but also F_R yields the negative sign of the second residuum and the correct asymptotic behaviour for $p^2 \to \infty$.

We suppressed the cuts from the mass zero poles of F and G, hence F_R has only the poles discussed. A physical 3-particle cut from $p^2 = 9\varkappa^2$ will appear only after dressing the Green's function to a physical pole for $p^2 = \varkappa^2$ in higher approximations, where we expect $G \Rightarrow F$, that means the pole theory gets one propagator only.

I wish to thank Prof. H. STUMPF for many helpful discussions about functional quantum theory.

Appendix

1. Canonical formulation

The functional Eq. (3) corresponds to a nonlinear field equation for the eight-component Hermitean Weyl-spinors isospinors ψ

$$D_{\alpha\beta} \psi_\beta = V_{\alpha\beta\gamma\delta} : \psi_\beta \psi_\gamma \psi_\delta : . \qquad (A1)$$

D and V are defined in (4) resp. (5) of the text. The colons indicate the Wick product of the operators, i.e. the time-ordered vacuum expectation values are subtracted

$$V_{\alpha\beta\gamma\delta} : \psi_\beta \psi_\gamma \psi_\delta : = V_{\alpha\beta\gamma\delta}[\psi_\beta \psi_\gamma \psi_\delta - 3 F_{\beta\gamma} \psi_\delta] . \ (A2)$$

Both theories may be reformulated in terms of canonical fields. That does not lead to any new physical consequences, but we get some better inside into the structure of the theories. Particularly, we may understand the different role of the indefinite metric in both cases.

Let us consider the pole theory first. The spinor potentials satisfy noncanonical commutation relations

$$\{\psi_\alpha^{\mathrm{p}}, \psi_\beta^{\mathrm{p}}\}_{t_\alpha = t_\beta} = \left\{ \frac{\partial}{\partial t_\alpha} \psi_\alpha^{\mathrm{p}}, \psi_\beta^{\mathrm{p}} \right\}_{t_\alpha = t_\beta} = 0 , \qquad (A3)$$

$$\left\{ \frac{\partial}{\partial t_\alpha} \psi_\alpha^{\mathrm{p}}, \frac{\partial}{\partial t_\beta} \psi_\beta^{\mathrm{p}} \right\}_{t_\alpha = t_\beta} = \varrho_0 \, \delta_{\alpha\beta} \, \delta^3(\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta) .$$

Differentiating ψ^{p} we obtain a field χ^{p} satisfying canonical commuting relations

$$\chi_\alpha^x := \frac{1}{i} \, D_{\alpha\beta}^1 \psi_\beta^{\mathrm{p}} = \Gamma_{\alpha\beta}^\nu \, \delta(x_\alpha - x_\beta) \frac{\delta}{\delta x_\beta^\nu} \psi_\beta^{\mathrm{p}}, \ (A4)$$

$$\{\chi_\alpha^{\mathrm{p}}, \chi_\beta^{\mathrm{p}}\}_{t_\alpha = t_\beta} = \varrho_0 \, \delta_{\alpha\beta} \, \delta^3(\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta) . \qquad (A5)$$

Neglecting possible inhomogeneous terms we get from (A1) a field equation of first order for the canonical field

$$\overline{D_{\alpha\beta}^1} \, \chi_\beta^{\mathrm{p}} = \overset{\mathrm{c}}{V}_{\alpha\beta\gamma\delta}^{\mathrm{p}} : \chi_\beta^{\mathrm{p}} \chi_\gamma^{\mathrm{p}} \chi_\delta^{\mathrm{p}} : . \qquad (A6)$$

$\overline{D^1}$ is the operator D^1 (A4) with $\overline{\Gamma}$ instead of Γ, $\overset{\mathrm{c}}{V}^{\mathrm{p}}$ is a nonlocal vertex

$$\overset{\mathrm{c}}{V}_{\alpha\beta\gamma\delta}^{\mathrm{p}} := \overset{\mathrm{c}}{G}_{\alpha\alpha'} \overset{\mathrm{c}}{G}_{\beta\beta'} \overset{\mathrm{c}}{G}_{\gamma\gamma'} \overset{\mathrm{c}}{G}_{\delta\delta'} \, V_{\alpha'\beta'\gamma'\delta'} \qquad (A7)$$

with the canonical Green's function

$$\overset{\mathrm{c}}{G}_{\alpha\beta} := i(D^1)_{\alpha\beta}^{-1} = i \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\Gamma_{\alpha\beta}^\nu p_\nu}{p^2} \, e^{-i(p, x_\alpha - x_\beta)} \ (A8)$$

The contraction function $\overset{\mathrm{c}}{F}$ has its usual canonical form

$$F_{\alpha\beta}^{\mathrm{p}} := \langle 0 \mid T \chi_\alpha^{\mathrm{p}} \chi_\beta^{\mathrm{p}} \mid 0 \rangle \qquad (A9)$$
$$= i \int \mathrm{d}m^2 \, \varrho(m^2) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\Gamma_{\alpha\beta}^\nu p_\nu}{p^2 - m^2} \, e^{-i(p, x_\alpha - x_\beta)} .$$

Comparing these formulae with the respective ones for ψ, all Γ and $\overline{\Gamma}$ are exchanged.

Up to this point you may treat Heisenberg's dipole theory similarly with

$$\chi_\alpha^{\mathrm{d}} := - \frac{\partial}{\partial x_\alpha^\mu} \frac{\partial}{\partial x_{\alpha\mu}} \psi_\alpha^{\mathrm{d}} . \qquad (A10)$$

The χ^{d} again satisfy the canonical commutation relations (A5)[24] and a field equation of first order

$$D_{\alpha\beta}^1 \chi_\beta^{\mathrm{d}} = \overset{\mathrm{c}}{V}_{\alpha\beta\gamma\delta}^{\mathrm{d}} : \chi_\beta^{\mathrm{d}} \chi_\gamma^{\mathrm{d}} \chi_\delta^{\mathrm{d}} : . \qquad (A11)$$

But in this case $\overset{\mathrm{c}}{V}^{\mathrm{d}}$ is an unsymmetrical vertex. In momentum space we obtain

$$\overset{\mathrm{c}}{V}_{\alpha\beta\gamma\delta}^{\mathrm{d}} = \frac{p_\alpha^2}{p_\beta^2 p_\gamma^2 p_\delta^2} \, V_{\alpha\beta\gamma\delta} . \qquad (A12)$$

The unsymmetrical form of $\overset{\mathrm{c}}{V}^{\mathrm{d}}$ reflects the different behaviour of G and F for big momenta. In contrary to the pole theory the canonical vertex (A12) contains p^2 in the numerator, i.e. a differential operator in space-time, destroying the canonical character of the theory again. To understand this let us apply (A11) to a time-ordered product of two χ^{d} for example

$$D_{\alpha\beta}^1 T \chi_\beta^{\mathrm{d}} \chi_\mu^{\mathrm{d}} = T D_{\alpha\beta}^1 \chi_\beta^{\mathrm{d}} \chi_\mu^{\mathrm{d}} + i \, \delta(t_\alpha - t_\mu) \{\chi_\alpha^{\mathrm{d}}, \chi_\mu^{\mathrm{d}}\}$$
$$= T \overset{\mathrm{c}}{V}_{\alpha\beta\gamma\delta}^{\mathrm{d}} : \chi_\beta^{\mathrm{d}} \chi_\gamma^{\mathrm{d}} \chi_\delta^{\mathrm{d}} : \chi_\mu^{\mathrm{d}} + i \, \varrho_0 \, \delta_{\alpha\mu} . \qquad (A13)$$

Here we have the ϱ_0-term from the canonical commutation relations (A5) for χ^{d}. But to derive an equation for the τ-functions we have to commute the time-ordering operator T with the vertex. This is possible with the original form (4) of V and with the nonlocal vertex (A7) of the pole theory too,

since they contain no derivatives. But because of the factor p_α^2 in (A12), $\overset{c}{V}{}^{d}$ does not commute with T, and a careful investigation shows that we pick up another ϱ_0-term compensating the ϱ_0-term in (A13):

$$D_{\alpha\beta}^1 \, T \, \chi_\beta^d \, \chi_\mu^d = \overset{c}{V}{}^{d}_{\alpha\beta\gamma\delta} \, T : \chi_\beta^d \, \chi_\gamma^d \, \chi_\delta^d : \chi_\mu^d \,. \quad (A14)$$

Therefore we get a functional equation again without any ϱ_0-term i.e. an equation like (3) with $r=0$ in spite of the canonical commutation relations: One cannot change the character of a theory by mere reformulation. This pseudocanonical form of the dipole theory as well as the noncanonical form allows no simple calculation of the fermion propagator.

In the case of the new pole theory, however, the ϱ_0-term is not compensated and we have a canonical theory with a special nonlocal interaction, the poles of which do not correspond to physical particles. Therefore the selfconsistence considerations of Chapter 5 are successfull.

2. *Symmetrical τ-function systems*

To illustrate the functional average methods of Chapter 3 the corresponding symmetrized τ-equations are given in a graphical representation[14].

$$\text{(A15)}$$

In the case of the momentum average we use further

$$\text{(A16)}$$

The functional Eq. (11) is represented by a system of τ-equations, we will write down as an example the τ_3-equation:

$$\text{(A17)}$$

The more complicated φ-equation (13) is obtained from (A17) by contracting out all possible F (see[14]).

By the integral average (29) we simply obtain the symmetrized Eq. (A17), i.e.

$$\text{(A18)}$$

Before writing down the equation for the momentum average, we rewrite the momentum subsidiary condition (16)

$$\text{(A19)}$$

In the case of the momentum average hence from the τ_n-function originate n terms, each with a factor p^μ at one leg. That gives the factors 3 and 2 in (48) resp. (59). The momentum average (22) requires the iteration of exactly that leg the factor p^μ is standing at. This yields

$$\text{(A20)}$$

The mass average (21) may be illustrated in a similar manner. Because of the two-fold iteration required the equations are more complicated and we do not write them down.

The approximation method (43) yields the eigenvalue equations

$$\text{(A21)}$$

$$\text{(A22)}$$

The selfconsistence Eq. (94) reads

$$\text{(A23)}$$

3. *Calculation of the integrals*

3.1. Fermion integrals

The twofold convolution integral (54) diverges for $r=1$, $e=1$ logarithmically. We have to calculate the other three. They certainly exist for

$$\lambda = I^2/\varkappa^2 < 0 \,,$$

since they have no singularities in this region. For $\lambda \geqq 0$ branch points occur for $\lambda = 0, 1, 4$. We will not discuss the complicated structure of the corresponding Riemann surfaces [26]. The physical sheet, however, is reached from $\lambda < 0$ by analytic continuation. If one extends all cuts along the real axis to $+\infty$, the physical sheet is defined by the usual $i\varepsilon$-prescription in the denominators. All this branch points arise from unphysical mass zero particles, hence we suppress the corresponding imaginary parts. Then the integrals are not analytic in the whole upper or lower halfplane. For possible consequences of such nonanalytic singularities to the causal structure of the theory we refer to the literature [16].

In our approximative calculations there are no physical cuts at all, hence we may use the simple prescription: Calculate all integrals with Feyman's-$i\varepsilon$ and take the real part afterwards. This yields the absolute values in the logarithms of the q_i and L functions.

For explicit calculations we join denominators by

$$\frac{1}{a^{n+1}} \frac{1}{b^{n+1}} = \frac{(n+m+1)!}{n!\,m!} \int_0^1 dx \, \frac{x^n (1-x)^m}{[ax+b(1-x)]^{n+m+2}} \qquad (A\,24)$$

and remove momenta in the numerators by

$$\frac{(p-q)^\mu}{[(p-q)^2 - m^2]^n} = \frac{\partial}{\partial a_\mu} \frac{1}{n+1} \frac{1}{[(p-q)^2 - (a, p-q) - m^2]^{n+1}}\bigg|_{a_\mu = 0}. \qquad (A\,25)$$

The arising integrals have less powers of momenta in the denominators than the original integrals, hence they may diverge. They are calculated by

$$\int \frac{d^4 p}{p^2 - \alpha} = -i\pi^2 \alpha \ln \alpha + c_0 + c_1 \alpha \qquad (A\,26)$$

since the indefinite constants disappear after the differentiation (A 25). Finally one has to evaluate the parameter integrals (A 24).

The integral average yields (54)

$$\frac{(\varkappa^2)^{2-r}}{\pi^4} \int d^4 r \, d^4 s \, \frac{(I-r-s)^\mu r^\nu s^\varrho}{[(I-r-s)^2]^{1+r}[r^2 - \varkappa^2] \, [r^2]^{2-r}[s^2 - \varkappa^2][s^2]^{2-r}} =: L_g (g^{\mu\nu} I^\varrho + g^{\mu\varrho} I^\nu + g^{\nu\varrho} I^\mu) + L_I \frac{I^\mu I^\nu I^\varrho}{I^2}, \qquad (A\,27)$$

$$L_I = 2\lambda \frac{\partial}{\partial\lambda} L_g, \qquad L_g = \frac{1}{2} \frac{\partial}{\partial\lambda} L_0. \qquad (A\,28)$$

With (53) we obtain the fermion function L^1

$$L^1(\lambda) = \frac{1}{\lambda^r} \left(3 + \lambda \frac{\partial}{\partial\lambda}\right) \frac{\partial}{\partial\lambda} L_0(\lambda). \qquad (A\,29)$$

Finally, we have to calculate in the pole resp. dipole case

$$L_0^p(\lambda) = \int_0^1 dx \, dy \, dt \, dz \, t \ln[xz + y(1-z) - \lambda tz(1-z)], \qquad (A\,30)$$

$$L_0^d(\lambda) = \int_0^1 dx \, dy \, dt \, dz \, \frac{(1-x)(1-y)(1-t)}{xz + y(1-z) - \lambda tz(1-z)}. \qquad (A\,31)$$

Usually it is convenient not to calculate L_0 directly but to use (A 29) at a suitable point during the calculation [27]. In the dipole case, however, we need L_0 for the momentum average, too. In this case we have to evaluate

$$3 \frac{\varkappa^2}{\pi^4} \int d^4 r \, d^4 s \, \frac{(I-r-s)^\mu (I-r-s)^\nu r^\varrho s^\sigma}{(I-r-s)^2 [r^2 - \varkappa^2][r^2]^2[s^2 - \varkappa^2][s^2]^2}. \qquad (A\,32)$$

From (A 32) arise in an expansion analogous to (A 27) five different functions instead of two. But observing that (A 32) vanishes if contracted by $g_{\mu\nu}$ we obtain for the fermion function L^{dq} after some calculations

$$L^{dq}(\lambda) = \frac{3}{2\lambda} \left(1 + 3\lambda \frac{\partial}{\partial\lambda}\right) L_0^d(\lambda). \qquad (A\,33)$$

Before writing down the L-functions explicitly we introduce the functions

$$R_1(\lambda) := \sqrt{\lambda(\lambda - 4)} \, \ln\left(\frac{\sqrt{4-\lambda}+\sqrt{-\lambda}}{\sqrt{4-\lambda}-\sqrt{-\lambda}}\right)^2, \tag{A34}$$

$$R_2(\lambda) := \ln^2\left(\frac{\sqrt{4-\lambda}+\sqrt{-\lambda}}{\sqrt{4-\lambda}-\sqrt{-\lambda}}\right)^2, \tag{A35}$$

$$\mathfrak{L}_2(\lambda) := -\int_0^\lambda \frac{1}{t}\ln(1-t)\,\mathrm{d}t. \tag{A36}$$

λ should be replaced by $\lambda + i\varepsilon$ in (A34—36). We need the real parts on the physical sheet only and obtain after $\varepsilon \to +0$.

$$\mathrm{Re}\,R_1(\lambda) = 4\sqrt{|\lambda(\lambda-4)|}\left\{\theta(-\lambda)\,\mathrm{Arsh}\,\frac{\sqrt{-\lambda}}{2} - \theta(4\lambda-\lambda^2)\arcsin\frac{\sqrt{\lambda}}{2} - \theta(\lambda-4)\,\mathrm{Arch}\,\frac{\sqrt{\lambda}}{2}\right\}, \tag{A34$'$}$$

$$\mathrm{Re}\,R_2(\lambda) = 16\left\{\theta(-\lambda)\,\mathrm{Arsh}^2\,\frac{\sqrt{-\lambda}}{2} - \theta(4\lambda-\lambda^2)\arcsin^2\frac{\sqrt{\lambda}}{2} + \theta(\lambda-4)\left(\mathrm{Arch}^2\,\frac{\sqrt{\lambda}}{2} - \frac{\pi^2}{4}\right)\right\}, \tag{A35$'$}$$

$$\mathrm{Re}\,\mathfrak{L}_2(\lambda) = -\theta(-\lambda+1)\left(\frac{\pi^2}{6} + \frac{1}{2}\ln^2(-\lambda) + \sum_{\nu=1}^\infty \frac{1}{\nu^2}\frac{1}{\lambda^\nu}\right)$$
$$+ \theta(\lambda^2-1)\sum_{\nu=1}^\infty \frac{\lambda^\nu}{\nu^2} + \theta(\lambda-1)\left\{\frac{\pi^2}{3} - \frac{1}{2}\ln^2\lambda - \sum_{\nu=1}^\infty \frac{1}{\nu^2}\frac{1}{\lambda^\nu}\right\}. \tag{A36$'$}$$

In the pole case we obtain

$$L^{\mathrm{p}}(\lambda) = \frac{1}{6}\ln|\lambda| + \frac{(1-\lambda)^3}{3\lambda^3}\ln|1-\lambda| + \frac{4-\lambda}{12\lambda^2}\mathrm{Re}\,R_1(\lambda) + \frac{1}{3\lambda^2} \tag{A37}$$

and especially Eq. (65).

In the dipole case we obtain

$$L_0^{\mathrm{d}}(\lambda) = \frac{1}{360}\Bigg[\left(-\frac{30}{\lambda^2} + \frac{40}{\lambda} - \frac{47}{2} - 7\lambda + \frac{\lambda^2}{4}\right)\mathrm{Re}\,R_1(\lambda)$$
$$+ \left(\frac{14}{\lambda^2} - \frac{91}{\lambda} - 11 + 29\lambda - \lambda^2\right)(1-\lambda)\ln|1-\lambda| - \frac{46}{\lambda} + 47 - \lambda$$
$$+ 3\left(\frac{5}{\lambda^2} - \frac{15}{2\lambda} + 5 - \frac{5}{2}\lambda\right)\mathrm{Re}\,R_2(\lambda) + 60(2-\lambda)\left(\mathrm{Re}\,\mathfrak{L}_2(\lambda) + \frac{\pi^2}{6}\right) \tag{A38}$$
$$+ \left(-60\lambda + 15\lambda^2 - \tfrac{1}{2}\lambda^3\right)\ln|\lambda|\Bigg]$$

and with (A29) resp. (A33) the fermion functions

$$L^{\mathrm{di}}(\lambda) = \frac{1}{48\lambda^2}\left(-6 - \lambda - 7\lambda^2 + \frac{\lambda^3}{2}\right)\mathrm{Re}\,R_1(\lambda) + \frac{1-\lambda^2}{16\lambda^2}\mathrm{Re}\,R_2(\lambda)$$
$$+ \frac{1}{24\lambda^2}(7 - 9\lambda + 15\lambda^2 - \lambda^3)(1-\lambda)\ln|1-\lambda| + \frac{1-\lambda}{24\lambda}$$
$$+ \frac{1}{48}(-24 + 16\lambda - \lambda^2)\ln|\lambda| - \frac{1}{2}\left(\mathrm{Re}\,\mathfrak{L}_2(\lambda) + \frac{\pi^2}{6}\right) \tag{A39}$$
$$= -\frac{1}{2}\ln|\lambda| + \frac{1}{2} - \frac{\pi^2}{12} + \frac{\lambda}{3}\ln\lambda - \frac{4}{9}\lambda + O(\lambda^2) = \frac{1}{4\lambda^2}\left(\ln^2\lambda + \ln\lambda - \pi^2 + \frac{5}{4} + \cdots\right)$$

$$L^{\mathrm{dq}}(\lambda) = \frac{1}{96\lambda^2}\left(\frac{60}{\lambda} - 26 - 16\lambda - 19\lambda^2 + \lambda^3\right)\mathrm{Re}\,R_1(\lambda)$$
$$+ \frac{1}{48}\left(-\frac{14}{\lambda^3} + \frac{28}{\lambda^2} - \frac{28}{\lambda} + 40 - 2\lambda\right)(1-\lambda)\ln|1-\lambda|$$
$$+ \frac{1}{16}\left(-\frac{5}{\lambda^3} + \frac{3}{\lambda^2} + \frac{1}{\lambda} - 2\right)\mathrm{Re}\,R_2(\lambda) + \frac{1-2\lambda}{2\lambda}\left(\mathrm{Re}\,\mathfrak{L}_2(\lambda) + \frac{\pi^2}{6}\right)$$
$$+ \frac{1}{48} - 48 + 21\lambda - \lambda^2)(\ln|\lambda| + \frac{1}{48}\left(\frac{46}{\lambda^2} + \frac{16}{\lambda} - 2\right) \tag{A40}$$
$$= \frac{1}{\lambda}\left(\frac{\pi^2}{12} + \frac{1}{4}\right) - \ln\lambda + \frac{19}{12} - \frac{\pi^2}{6} + \frac{7}{16}\lambda\ln\lambda + O(\lambda)$$
$$= \frac{1}{4\lambda^2}\left(3\ln^2\lambda - 6\ln\lambda - 3\pi^2 - \frac{3}{4} + \cdots\right).$$

The fermion function $L^{\mathrm{dF}}(\lambda)$ of the F-average is obtained from

$$L^{\mathrm{dF}}(\lambda) = 2\,L^{\mathrm{di}}(\lambda) - L^{\mathrm{dq}}(\lambda)\,. \tag{A41}$$

We will not write it down explicitly.

3.2. Boson integrals

For the integral average we have to calculate (62) with $e = 0$.

$$i\,\frac{4\,\varkappa^2}{\pi^2} \int \mathrm{d}^4 q \,\frac{(I-q)^\alpha q^\beta}{[(I-q)^2]^{1+r}[q^2-\varkappa^2][q^2]^{2-r}} =: K_g^i\,g^{\alpha\beta} + K_I^i\,\frac{I^\alpha I^\beta}{I^2}\,. \tag{A42}$$

With the parameter integral

$$K_g^i(\lambda) = -2 \int_0^1 \mathrm{d}x\,\mathrm{d}y\,\frac{y}{x-\lambda y}\left[\frac{(1-x)(1-y)}{y}\right]^{1-r},\qquad K_I^i = 2\,\lambda\,\frac{\partial}{\partial\lambda}\,K_g^i \tag{A43}$$

we express the boson functions of the integral average:

$$q_0^i(\lambda) = \left(1 - \lambda\,\frac{\partial}{\partial\lambda}\right) K_g^i(\lambda),\qquad q_1^i(\lambda) = \left(1 + \lambda\,\frac{\partial}{\partial\lambda}\right) K_g^i(\lambda)\,. \tag{A44}$$

Explicite evaluation of (A43) gives

$$K_g^{\mathrm{pi}} = \ln(-\lambda) + \frac{1-\lambda^2}{\lambda^2}\ln(1-\lambda) + \frac{1}{\lambda}\,, \tag{A45}$$

$$K_g^{\mathrm{di}} = \left(1 - \frac{\lambda}{3}\right)\ln(-\lambda) - \frac{(1-\lambda)^3}{3\,\lambda^2}\ln(1-\lambda) - \frac{1}{3\,\lambda} + \frac{1}{3}\,, \tag{A46}$$

and from the real parts of (A45) resp. (A46) we obtain the formulae (68), (69) resp. (79), (80).

For the momentum average we proceed similarly. We have to calculate (63) with $e = 1$.

$$2\,i\,\frac{4\,\varkappa^2}{\pi^2} \int \mathrm{d}^4 q \,\frac{q^\alpha (I-q)^\beta (I-q)^\gamma}{[(I-q)^2]^{1+r}[q^2-\varkappa^2][q^2]^{2-r}} =: K_0(g^{\alpha\beta} I^\gamma + g^{\alpha\gamma} I^\beta) + K_1 g^{\beta\gamma} I^\alpha + K_2\,\frac{I^\alpha I^\beta I^\gamma}{I^2}\,. \tag{A47}$$

With the parameter integrals

$$K_0(\lambda) = -4 \int_0^1 \mathrm{d}x\,\mathrm{d}y\,\frac{y(1-y)}{x-\lambda y}\left[\frac{(1-x)(1-y)}{y}\right]^{1-r},\qquad K_2 = 2\,\lambda\,\frac{\partial}{\partial\lambda}\,K_0\,. \tag{A48}$$

$$K_1(\lambda) = +4 \int_0^1 \mathrm{d}x\,\mathrm{d}y\,\frac{y^2}{x-\lambda y}\left[\frac{(1-x)(1-y)}{y}\right]^{1-r} \tag{A49}$$

we express the boson functions of the momentum average:

$$q_0^{\mathrm{q}} = \tfrac{1}{2}(K_0 - K_1 - K_2),\qquad q_1^{\mathrm{q}} = 2\,K_0 - q_0\,. \tag{A50}$$

To obtain the explicit formulae (74), (75) for the pole and (82), (83) for the dipole theory, we evaluate the parameter integrals:

$$K_0^{\mathrm{p}} = \frac{2}{3}\ln(-\lambda) - \frac{2}{3\,\lambda^3}(2+\lambda)(1-\lambda)^2\ln(1-\lambda) - \frac{4}{3\,\lambda^2} + \frac{4}{3\,\lambda}\,, \tag{A51}$$

$$K_1^{\mathrm{p}} = -\frac{4}{3}\ln(-\lambda) - \frac{4}{3\,\lambda^3}(1-\lambda^3)\ln(1-\lambda) - \frac{4}{3\,\lambda^2} - \frac{2}{3\,\lambda}\,, \tag{A52}$$

$$K_0^{\mathrm{d}} = \left(\frac{4}{3} - \frac{\lambda}{3}\right)\ln(-\lambda) + \frac{(1-\lambda)^4}{3\,\lambda^3}\ln(1-\lambda) + \frac{1}{3\,\lambda^2} - \frac{7}{6\,\lambda} + \frac{1}{3}\,, \tag{A53}$$

$$K_1^{\mathrm{d}} = \left(-\frac{2}{3} + \frac{\lambda}{3}\right)\ln(-\lambda) + \frac{(1+\lambda)(1-\lambda)^3}{3\,\lambda^3}\ln(1-\lambda) + \frac{1}{3\,\lambda^2} - \frac{1}{2\,\lambda} - \frac{1}{3}\,. \tag{A54}$$

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On the Functional Definition and Calculation of Global Observables in Nonlinear Spinor Field Quantum Theory

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Nonlinear spinor theory contains unobservable field operators which cannot be identified with free field operators. Therefore for the comparison with experiment a theory of observables for nonlinear spinor fields is required. This theory is developed for global observables by means of a map into functional space, and leads to a functional quantum theory of nonlinear spinor fields.

Nonlinear spinor field theory with noncanonical relativistic Heisenberg quantization is an approach to a unified microscopic description of matter[1,2]. Therefore, in general its spinor field operators $\Psi_\alpha(x)$ cannot be identified with a special free matter field of conventional Lagrangian coupling field theories, i. e. the spinor field operators are unobservable quantities in a more general way than is assumed usually. As the physical observables have to be defined with respect to the observable quantities of real matter, like asymptotic free particles etc., they can be represented simply only by field operators having a direct physical interpretation. As this is not the case with the nonlinear spinor field, one has to expect therefore a rather complicated connection between these observables and the operators $\Psi_\alpha(x)$. From this follows that the methods of construction of observables provided by coupling theories, and even in the wider sense by conventional quantum

field theory, do not suffice to solve this task correctly for nonlinear spinor theory. This is shown in detail in[3] and has been ignored so far. The ignorance of this fact is one of the reasons why dynamical calculations in this theory have not led to a considerable progress in the last decade. Therefore a special theory of observables is required to obtain meaningfull physical informations from nonlinear spinor theory. Theoretically, the set of observables is given by local and global quantities. Global observables are defined by the maximal set of simultaneously diagonalizable group generators of the corresponding symmetry groups and by the S-matrix. Local observables are functionals of the field operators on the position x. As all observations in microphysics are done really by scattering processes, local observables are not required in principle[4]. Therefore, for a high energy quantum theory the definition and use of global observables is sufficient. In the following we develop therefore a theory only of global observables for the nonlinear spinor field. It will be shown that such a theory is provided by